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ON A METHOD OF SOLVING INTEGRAL EQUATIONS AND ITS APPLICATION TO THE PROBLEM OF THE BENDING OF A PLATE WITH A CRUCIFORM INCLUSION*

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A scheme for a method of solving a certain integral equation is proposed consisting of the construction of a system of solutions of the characteristic equation with a complete system of special right-hand sides (polynomials, say) and writing the approximate solution of the initial equation in the form of a linear combination of the functions constructed. The scheme is realized in the case when the characteristic equation is reduced to a Mellin convolution and is solved exactly by the factorization method using the Mellin transform. In particular, such an approach provides an efficient solution of an integral equation with a fixed singularity in the kernel. A system of two integral equations obtained in the problem of the bending of a rectangular plate with a cruciform inclusion is solved as an illustration. In the case of branches of the inclusion of identical length the system is solved by the ordinary factorization method, in the case of branches of the inclusion of different length the factorization method is modified substantially.

1. The general scheme of the method of basis right-hand sides. We consider an integral equation of the form

$$\int_0^1 (L(t,\tau)+K(t,\tau))\varphi(\tau)\,d\tau - f(t) = 0 \quad (0\leqslant t\leqslant 1) \tag{1.1}$$

where $L(t,\tau)$ is the characteristic part of the kernel that includes all the singularities existing in the kernel and $K(t,\tau)$ is the regular part of the kernel. It is assumed that for the characteristic equation

$$\int_0^1 L(t,\tau)\varphi_0(\tau)\,d\tau = g(t) \quad (0\leqslant t\leqslant 1) \tag{1.2}$$

an exact solution can be constructed. The properties of the functions $\varphi(\tau)$ and $\varphi_0(\tau)$ (the arrangement and nature of the singularities) are in agreement here.

Assuming that the functions $K(t,\tau)$ and $f(t)$ in (1.1) are sufficiently smooth and approximated well by polynomials, and following /1/ (p.9) we consider a system of functions $\theta_m^{\pm}(\tau)$ that are solutions of the equations

$$\int_0^1 \begin{Bmatrix} L(t,\tau) \\ L(\tau,t) \end{Bmatrix} \theta_m^{\pm}(\tau)\,d\tau = t^m \quad (0\leqslant t\leqslant 1,\ m=0,1,2,\ldots) \tag{1.3}$$

(for certain kernels $L(t,\tau)$ the range of variation of m can be narrowed, see Sect.3). We construct the function (/2/, p.40)

$$\| \pi_n^{\pm}(\tau), p_n^{\mp}(t) \| = \sum_{l=0}^{n} a_{nl}^{\pm} \| \theta_l^{\pm}(\tau), t^l \| \tag{1.4}$$

for which

$$\int_0^1 \begin{Bmatrix} L(t,\tau) \\ L(\tau,t) \end{Bmatrix} \pi_n^{\pm}(\tau)\,d\tau = \sigma_n^{\pm} p_n^{\mp}(t) \quad (0\leqslant t\leqslant 1,\ n=0,1,2,\ldots) \tag{1.5}$$

and the biorthogonality conditions are satisfied

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$$\int_0^1 p_n^{\pm}(t)\,\pi_m^{\pm}(t)\,dt = N_n^{\pm}\delta_{mn} \tag{1.6}$$

Formulas (1.5) and (1.6) enable us to write the exact solution of (1.2) in the following form (analogous to /1/, p.7)

$$\varphi(\tau) = \sum_{n=0}^{\infty} \frac{1}{N_n^{-}\sigma_n^{+}} \left(\int_0^1 g(t)\,\pi_n^{-}(t)\,dt\right)\pi_n^{+}(\tau) \tag{1.7}$$

and also enable us to use a method analogous to the method of orthogonal polynomials /1, 2/ for the approximate solution of (1.1):

a) write the desired function in the form

$$\varphi(\tau) = \sum_{n=0}^{N} \Phi_n \pi_n^{+}(\tau) \tag{1.8}$$

b) determine Φ_n from the condition of the orthogonality of the left-hand side of (1.1) to the functions $\pi_m^{-}(t)$:

$$N_m^{-}\sigma_m^{+}\Phi_m + \sum_{n=0}^{N} d_{mn}\Phi_n = F_m \quad (m = 0, 1, \ldots, N) \tag{1.9}$$

$$d_{mn} = \int_0^1\int_0^1 K(t,\tau)\,\pi_n^{+}(\tau)\,\pi_m^{-}(t)\,d\tau\,dt, \quad F_m = \int_0^1 f(t)\,\pi_m^{-}(t)\,dt$$

The function (1.8) can also be written in the form

$$\varphi(\tau) = \sum_{n=0}^{N} \varphi_n \theta_n^{+}(\tau) \tag{1.10}$$

by determining φ_n from the condition of orthogonality of the left-hand side of (1.1) to the functions $\theta_m^{-}(t)$

$$\sum_{n=0}^{N} (a_{mn} + b_{mn})\,\varphi_n = f_m \quad (m = 0, 1, \ldots, N) \tag{1.11}$$

$$a_{mn} = \int_0^1 t^n \theta_m^{-}(t)\,dt, \quad f_m = \int_0^1 f(t)\,\theta_m^{-}(t)\,dt$$

$$b_{mn} = \int_0^1\int_0^1 K(t,\tau)\,\theta_n^{+}(\tau)\,\theta_m^{-}(t)\,d\tau\,dt$$

Utilizing (1.10) and (1.11) in place of (1.8) and (1.9) enables us to dispense with the tedious construction of the biorthogonal systems (1.4) in the general case.

If the characteristic part of the matrix-**kernel** is diagonal

$$\int_0^1 \Big(L_\alpha(t,\tau)\,\varphi_\alpha(\tau) + \sum_{\beta=1}^{M} K_{\alpha\beta}(t,\tau)\,\varphi_\beta(\tau)\Big)\,d\tau - f_\alpha(t) = 0 \tag{1.12}$$

$$(0 \leqslant t \leqslant 1,\ \alpha = 1, 2, \ldots, M)$$

then we have in place of (1.3), (1.10) and (1.11)

$$\int_0^1 \begin{Bmatrix} L_\alpha(t,\tau) \\ L_\alpha(\tau,t) \end{Bmatrix} \theta_{\alpha m}^{\pm}(\tau)\,d\tau = t^m, \quad \varphi_\alpha(\tau) = \sum_{n=0}^{N} \varphi_{\alpha n}\theta_{\alpha n}^{+}(\tau) \tag{1.13}$$

$$\sum_{\beta=1}^{M}\sum_{n=0}^{N} (a_{\alpha\beta mn} + b_{\alpha\beta mn})\,\varphi_{\beta n} = f_{\alpha m}$$

$$(\alpha = 1, 2, \ldots, M;\ m = 0, 1, \ldots, N)$$

$$a_{\alpha\beta mn} = \delta_{\alpha\beta}\int_0^1 t^n \theta_{\alpha m}^{-}(t)\,dt, \quad f_{\alpha m} = \int_0^1 f_\alpha(t)\,\theta_{\alpha m}^{-}(t)\,dt$$

$$b_{\alpha\beta mn}=\int_0^1\int_0^1 K_{\alpha\beta}(t,\tau)\,\theta_{\beta n}^{+}(\tau)\,\theta_{\alpha m}^{-}(t)\,d\tau\,dt$$

For a system of general form

$$\sum_{j=1}^{M}\int_0^{\lambda_j}(L_{ij}(t,\tau)+K_{ij}(t,\tau))\,\varphi_j(\tau)\,d\tau-f_i(t)=0 \qquad (1.14)$$
$$(0\leqslant t\leqslant\lambda_i,\ i=1,2,\ldots,M)$$

we obtain the following scheme

$$\sum_{j=1}^{M}\int_0^{\lambda_j}\begin{Bmatrix}L_{ij}(t,\tau)\\ L_{ji}(\tau,t)\end{Bmatrix}\theta_{j\beta n}^{\pm}(\tau)\,d\tau=p_{i\beta}{}^{\mp}t^n \qquad (1.15)$$
$$(0\leqslant t\leqslant\lambda_i;\ i=\overline{1,M};\ \beta=\overline{1,M};\ n=0,1,2,\ldots)$$

$$\varphi_j(\tau)=\sum_{\beta=1}^{M}\sum_{n=0}^{N}\varphi_{\beta n}\theta_{j\beta n}^{+}(\tau)$$
$$\sum_{\beta=1}^{M}\sum_{n=0}^{N}(a_{\alpha\beta mn}+b_{\alpha\beta mn})\,\varphi_{\beta n}=f_{\alpha m} \qquad (1.16)$$
$$(\alpha=1,2,\ldots,M;\ m=0,1,\ldots,N)$$
$$a_{\alpha\beta mn}=\sum_{i=1}^{M}\int_0^{\lambda_i}p_{i\beta n}(t)\,\theta_{i\alpha m}^{-}(t)\,dt,\quad f_{\alpha m}=\sum_{i=1}^{M}\int_0^{\lambda_i}f_i(t)\,\theta_{i\alpha m}^{-}(t)\,dt$$
$$b_{\alpha\beta mn}=\sum_{i=1}^{M}\sum_{j=1}^{M}\int_0^{\lambda_i}\int_0^{\lambda_j}K_{ij}(t,\tau)\,\theta_{j\beta n}^{+}(\tau)\,\theta_{i\alpha m}^{-}(t)\,d\tau\,dt$$

System (1.16) in $\varphi_{\beta n}$ is obtained from the condition of the orthogonality of the residual vector $\| d_1(t), d_2(t),\ldots, d_M(t)\|$ on the left-hand side of (1.14) to the vectors $\| \theta_{1\alpha m}^{-}(t), \theta_{2\alpha m}^{-}(t), \ldots, \theta_{M\alpha m}^{-}(t)\|$:

$$\sum_{i=1}^{M}\int_0^{\lambda_i}d_i(t)\,\theta_{i\alpha m}(t)\,dt=0$$
$$(\alpha=1,2,\ldots,M;\ m=0,1,\ldots,N)$$

The realizability of the scheme described is determined completely by the possibility of constructing the functions $\theta_m^{\pm}(\tau)$ in (1.3) in a form enabling a_{mn}, b_{mn} and f_m in (1.11) to be evaluated. This is done especially easily if the integral operators in (1.3) are Mellin convolutions

$$\int_0^1 g_{\pm}\left(\frac{t}{\tau}\right)\theta_m{}^{\pm}(\tau)\frac{d\tau}{\tau}=t^m \quad (0\leqslant t\leqslant 1) \qquad (1.17)$$

(or are reduced to such a form by using differentiation, say). Eqs.(1.17) can be solved by the factorization method (see below) whereupon the Mellin transforms are found

$$\Psi_m{}^{\pm}(p)=\int_0^1\theta_m{}^{\pm}(\tau)\,\tau^{p-1}\,d\tau \qquad (1.18)$$

and by the inversion formula, the functions $\theta_m^{\pm}(\tau)$ themselves.

By virtue of (1.11) and (1.18) $a_{mn}=\Psi_m{}^{-}(n+1)$. If the functions $K(t,\tau)$ and $f(t)$ are here expanded in power series

$$f(t)=\sum_{k=0}^{\infty}A_kt^k,\quad K(t,\tau)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}B_{kl}t^k\tau^l \qquad (1.19)$$

(in the general case in polynomial series), we obtain the following simple expressions:

$$f_m=\sum_{k=0}^{\infty}A_k\Psi_m{}^{-}(k+1),\quad b_{mn}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}B_{kl}\Psi_n{}^{+}(l+1)\,\Psi_m{}^{-}(k+1) \qquad (1.20)$$

The method described can be applied efficiently to equations with a fixed singularity in the kernel (for $t = \tau = 0$) which is obtained, in particular, in two-dimensional problems of elasticity theory with intersecting (branching) linear defects and linear defects emerging on the domain boundary /2-10/.

An investigation of such equations is carried out in /11/. In this case the main difficulty in solving (1.1) is associated with the complication in the behaviour of the function $\varphi(\tau)$ as $\tau \to 0$, which will be of the type $\tau^{-\gamma}$, as a rule, where γ is the root of a certain transcendental equation $G(\gamma) = 0$. Consequently, the desired function is written in the form $\varphi(\tau) = \tau^{\alpha}(1-\tau)^{\beta}\varphi_0(\tau)$, in /2-5/, where $\alpha = -\gamma_0$, γ_0 is the root of the equation $G(\gamma) = 0$ from a certain strip determined from the mechanical meaning of the problem. The function $\varphi_0(\tau)$ is approximated by polynomials or splines.

This method is fairly awkward even in the case of real γ_0 because of the difficulty of evaluating the integral

$$\int_0^1 (L(t,\tau) + K(t,\tau))\, \tau^{\alpha}(1-\tau)^{\beta}\varphi_0(\tau)\, d\tau \tag{1.21}$$

In the case of the presence of two complex-conjugate roots in the equation $G(\gamma) = 0$ the difficulties increase significantly. Moreover, the roots γ_k of the equation $G(\gamma) = 0$ with $\operatorname{Re}\gamma_k < \operatorname{Re}\gamma_0$, exert an influence on the smoothness of the function $\varphi_0(\tau)$ and the efficiency of its approximation by polynomials. In order to simplify the algorithm, $\alpha = \alpha_0$ is taken in /6-10/, the closest integer or half-integer to the right of γ_0, which enables simple quadrature formulas to be used to calculate (1.21). In substance, the same simplification is made in /12, 13/ also, where the approximate form of the mapping function results in smoothing out the angular points on the domain boundary when conformal mappings are utilized.

Being sufficiently simple algorithmically (see (1.11) and (1.20)), the proposed method simultaneously takes account of the behaviour of the desired function as $\tau \to 0$ most accurately since we have $\Psi_m^+(p) = R_m(p)[G(p)]^{-1}$ in (1.18) for $\operatorname{Re} p \leqslant \operatorname{Re}\gamma_0$, and by the inversion formula

$$\theta_m^+(\tau) = \frac{1}{2\pi i}\int_L \frac{R_m(p)}{G(p)}\tau^{-p}\,dp = \sum_k \operatorname*{Res}_{\operatorname{Re}\gamma_k \leqslant \operatorname{Re}\gamma_0}\left[\frac{R_m(p)}{G(p)}\right]\tau^{-\gamma_k} \tag{1.22}$$

i.e., all the roots of the equation $G(\gamma) = 0$ with $\operatorname{Re}\gamma \leqslant \operatorname{Re}\gamma_0$ are taken into account. Both these facts, as well as the new solution obtained below for the problem of the bending of a rectangular plate with a cruciform inclusion indicate the efficiency of the method.

We note that difficulties with the presence of non-integrable singularities of the contact forces as $\tau \to 1$ (analogous singularities are obtained in /2, 14-17/ in problems of the bending of plates with linear inclusions (in the form of line segments) must be overcome in the solution of the problem mentioned.

2. The problem of the bending of a rectangular plate with a cruciform inclusion. We consider a rectangular $(|x| \leqslant a_1 = a/2,\ |y| \leqslant b_1 = b/2)$ hinge-supported plate within which there is a thin absolutely rigid inclusion subjected to an applied force P of magnitude W_0 at the segments $y = 0,\ |x| \leqslant c_1 = c/2$ and $x = 0,\ |y| \leqslant d_1 = d/2$.

The problem is formulated as follows. Find the deflection of a plate satisfying the equation and boundary conditions

$$D\Delta^2 w = q(x, y) \quad (|x| \leqslant a_1,\ |y| \leqslant b_1) \tag{2.1}$$

$$w = M_x = 0 \quad (|x| = a_1,\ |y| \leqslant b_1) \tag{2.2}$$
$$w = M_y = 0 \quad (|y| = b_1,\ |x| \leqslant a_1)$$

as well as the conditions on the inclusion

$$w = W_0 \quad (y = 0,\ |x| \leqslant c_1 \quad \text{and} \quad x = 0,\ |y| \leqslant d_1) \tag{2.3}$$

$$w_y' = 0 \ (y = 0,\ |x| \leqslant c_1),\quad w_x' = 0 \ (x = 0,\ |y| \leqslant d_1) \tag{2.4}$$

The function $w(x, y)$ is obviously even in x and y.

We reduce problem (2.1)-(2.4) to a system of integral equations. To do this, we start, as in /2, 15-17/, from the fact that the presence of the inclusion causes a jump in the transverse forces

$$\psi_1(\xi) = V_y(\xi, -0) - V_y(\xi, +0),\quad \psi_2(\eta) = V_x(-0, \eta) - V_x(+0, \eta) \tag{2.5}$$

the functions ψ_1, ψ_2 are even, $\psi_1(\xi) \equiv 0$ for $c_1 < |\xi| < a_1$, $\psi_2(\eta) \equiv 0$ for $d_1 < |\eta| < b_1$.

Assuming the load is applied only to the inclusion, we write the right-hand side of (2.1) in the form

$$q(x,y)=\psi_1(x)\delta(y)+\psi_2(y)\delta(x)=$$
$$\frac{4}{ab}\sum_k{}'\sum_l{}'\cos\alpha x\cos\beta y\left(\int_{-c_1}^{c_1}\psi_1(\xi)\cos\alpha\xi\,d\xi+\int_{-d_1}^{d_1}\psi_2(\eta)\cos\beta\eta\,d\eta\right)$$
$$\alpha=\alpha_k=\pi a^{-1}k,\ \ \beta=\beta_l=\pi b^{-1}l$$

which results in the following expression for the function satisfying (2.1) and (2.2) in terms of the unknown functions $\psi_1(\xi)$, $\psi_2(\eta)$:

$$w(x,y)=\frac{4}{Dab}\sum_k{}'\sum_l{}'\frac{\cos\alpha x\cos\beta y}{(\alpha^2+\beta^2)^2}\left(\int_{-c_1}^{c_1}\psi_1(\xi)\cos\alpha\xi\,d\xi+\int_{-d_1}^{d_1}\psi_2(\eta)\cos\beta\eta\,d\eta\right)$$

Here and henceforth the prime on the summation sign means that summation is taken over all positive odd values of the variable mentioned.

Conditions (2.4) are satisfied because of the oddness of w_x' in x and w_y' in y and the uniform convergence of the series for w_x' and w_y' while conditions (2.3) result in a system of two integral equations in $\psi_1(\xi)$ and $\psi_2(\eta)$ $(\sigma=ba^{-1})$:

$$\int_{-c_1}^{c_1}\sum_k{}'\cos\alpha x\cos\alpha\xi\left(\sum_l{}'\frac{\sigma^3}{(l^2+\sigma^2k^2)^2}\right)\psi_1(\xi)\,d\xi+ \tag{2.6}$$
$$\int_{-d_1}^{d_1}\sum_k{}'\cos\alpha r\left(\sum_l{}'\frac{\sigma^3\cos\beta\eta}{(l^2+\sigma^2k^2)^2}\right)\psi_2(\eta)\,d\eta=\frac{DW_0}{4a^3}\pi^4\quad(|x|\leqslant c_1)$$

$$\int_{-c_1}^{c_1}\sum_l{}'\cos\beta y\left(\sum_k{}'\frac{\sigma^{-1}\cos\alpha\xi}{(k^2+\sigma^{-2}l^2)^2}\right)\psi_1(\xi)\,d\xi+$$
$$\int_{-d_1}^{d_1}\sum_l{}'\cos\beta y\cos\beta\eta\left(\sum_k{}'\frac{\sigma^{-1}}{(k^2+\sigma^{-2}l^2)^2}\right)\psi_2(\eta)\,d\eta=\frac{DW_0}{4a^3}\pi^4$$
$$(|y|\leqslant d_1)$$

We execute the following three transformations in system (2.6):

a) we make a change of variables and functions $x=c_1t$, $y=c_1t$, $\xi=c_1\tau$, $\eta=c_1\tau$, $\varphi_1(\tau)=\pi c_1\psi_1(\xi)$, $\varphi_2(\tau)=\pi c_1\psi_2(\eta)$, we introduce the notation $\varepsilon=ca^{-1}$, $\lambda=dc^{-1}$, $\gamma=\gamma_k=\pi\varepsilon k/2$ and we take account of the evenness of $\varphi_1(\tau)$ and $\varphi_2(\tau)$;

b) we sum the inner series by means of the formula

$$\sum_j{}'\frac{\cos(js)}{(j^2+a^2)^2}=\frac{\pi}{8a^3}\left((1+sa)e^{-sa}+\mu(a)\,\mathrm{ch}(sa)-\nu(a)\,sa\,\mathrm{sh}(sa)\right)$$
$$\mu(a)=\mathrm{th}\,\rho-1-\rho\,\mathrm{sech}^2\rho,\ \ \nu(a)=\mathrm{th}\,\rho-1,\ \ \rho=\pi a/2$$

resulting from formula 5.4.5.11-12 /18/;

c) we convert the slowly converging part of the outer series by using the formulas

$$\sum_k{}'\frac{\cos\gamma t\cos\gamma\tau}{k^3}=\frac{1}{2}\left(\frac{\pi\varepsilon}{4}\right)^2\Big[(t-\tau)^2\ln|t-\tau|+(t+\tau)^2\ln|t+\tau|+ \tag{2.7}$$
$$\left(2\ln\left(\frac{\pi\varepsilon}{4}\right)-3\right)(t^2+\tau^2)\Big]+$$
$$A_3-\pi^2\sum_{l=2}^{\infty}\left(\frac{\varepsilon}{2}\right)^{2l}(2l-3)!\,T_{2l-2}\sum_{k=0}^{l}\frac{t^{2l-2k}}{(2l-2k)!}\frac{\tau^{2k}}{(2k)!}$$
$$\sum_k{}'(1+\gamma\tau)\frac{\cos\gamma t}{k^3}e^{-\gamma\tau}=\left(1-\tau\frac{d}{d\tau}\right)\sum_k{}'\frac{\cos\gamma t}{k^3}e^{-\gamma\tau}=$$
$$\frac{1}{2}\left(\frac{\pi\varepsilon}{4}\right)^2\Big[(t^2+\tau^2)\ln(t^2+\tau^2)+\left(2\ln\left(\frac{\pi\varepsilon}{4}\right)-3\right)(t^2+\tau^2)+2\tau^2\Big]+$$
$$A_3-\pi^2\sum_{l=2}^{\infty}\left(\frac{\varepsilon}{2}\right)^{2l}(2l-3)!\,T_{2l-2}\sum_{k=0}^{l}\frac{t^{2l-2k}}{(2l-2k)!}\frac{\tau^{2k}}{(2k)!}(-1)^k(1-2k)$$

where /18/

$$A_3=\sum_k{}' k^{-3}\approx 1.0518,\quad T_s=\sum_{k=1}^{\infty}(-1)^k k^{-s}$$

Formulas (2.7) are derived in the same way as (1.5) /17/.

Consequently, the transformed system (2.6) is written in a form ensuring satisfaction of conditions (1.10)

$$\sum_{j=1}^{2}\int_0^{\lambda_j}\left[\frac{1}{2}\left(\frac{\pi\varepsilon}{4}\right)^2 L_{ij}(t,\tau)+K_{ij}(t,\tau)\right]\varphi_j(\tau)\,d\tau=\frac{DW_0}{a^2}\pi^4 \tag{2.8}$$

$$(0\leqslant t\leqslant\lambda_i)$$

$$i=1,2;\ \lambda_1=1,\ \lambda_2=\lambda;\ L_{12}(t,\tau)=L_{21}(t,\tau)=(t^2+\tau^2)\ln(t^2+\tau^2)$$

$$L_{11}(t,\tau)=L_{22}(t,\tau)=(t-\tau)^2\ln|t-\tau|+(t+\tau)^2\ln(t+\tau)$$

$$K_{ij}(t,\tau)=\frac{1}{2}\left(\frac{\pi\varepsilon}{4}\right)^2\left[\left(2\ln\frac{\pi\varepsilon}{4\sigma_i}-3\right)(t^2+\tau^2)+2(1-\delta_{ij})\tau^2\right]+ \tag{2.9}$$

$$\sigma_i^2A_3-\sigma_i^2\pi^2\sum_{l=2}^{\infty}\left(\frac{\varepsilon}{2\sigma_i}\right)^{2l}(2l-3)!\,T_{2l-2}\sum_{k=0}^{l}\frac{t^{2l-2k}}{(2l-2k)!}\times$$

$$\frac{\tau^{2k}}{(2k)!}[\delta_{ij}+(1-\delta_{ij})(-1)^k(1-2k)]+$$

$$\sigma_i^2\sum_k{}'\frac{1}{k^3}\left(\sum_{l=0}^{\infty}(-1)^l\left(\frac{\gamma}{\sigma_i}\right)^{2l}\frac{t^{2l}}{(2l)!}\right)\times$$

$$\left[\sum_{l=0}^{\infty}\left(\mu\left(\frac{\sigma}{\sigma_i^2}k\right)(1-2\delta_{ij})^l-\nu\left(\frac{\sigma}{\sigma_i^2}k\right)(1-\delta_{ij})2l\right)\left(\frac{\gamma}{\sigma_i}\right)^{2l}\frac{\tau^{2l}}{(2l)!}\right],$$

$$\sigma_1=1,\quad \sigma_2=\sigma$$

where expansions of the functions cos, cosh, sinh in power series are used in the last sum with respect to k.

System (2.8) is solved differently for identical $(\lambda=1)$ and different $(\lambda\neq1)$ lengths of the inclusion branches.

3. Inclusion branches of identical length $(\lambda=1)$. To reduce system (2.8) to the form (1.12) we make the change of functions $\varphi_{1,2}=\varphi_1{}^\circ\pm\varphi_2{}^\circ$ and form a new system of equations by taking the sum and difference of (2.8). We consequently obtain

$$\left(\frac{\pi\varepsilon}{4}\right)^2\int_0^1 L_\alpha(t,\tau)\varphi_\alpha{}^\circ(\tau)\,d\tau+\sum_{\beta=1}^{2}\int_0^1 M_{\alpha\beta}(t,\tau)\varphi_\beta{}^\circ(\tau)\,d\tau= \tag{3.1}$$

$$2\pi^4\frac{DW_0}{a^2}\delta_{1\alpha}$$

$$(0\leqslant t\leqslant 1),\ \alpha=1,2;\ L_\alpha=L_{11}-(-1)^\alpha L_{12}$$

$$M_{\alpha\beta}=\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{(i+1)(\alpha+1)+(j+1)(\beta+1)}K_{ij}$$

The kernels $L_\alpha(t,\tau)$ have mobile singularities (for $t=\tau$) and fixed singularities (for $t=\tau=0$) and are symmetric. Taking their evenness in t into account as well as the fact that $L_2(0,\tau)=0$, we consider instead of (1.13) the equations

$$\frac{1}{2\pi}\int_0^1 L_\alpha(t,\tau)\theta_{\alpha n}(\tau)\,d\tau=\frac{t^{2(n+\alpha)-2}}{(2(n+\alpha)-2)!}\equiv p_{\alpha n}(t) \tag{3.2}$$

$$(0\leqslant t\leqslant 1),\ n=0,1,2,\ldots;\ \alpha=1,2$$

We differentiate them thrice with respect to t to reduce them to the form (1.17)

$$\int_0^1 N_\alpha\left(\frac{t}{\tau}\right)\theta_{\alpha n}(\tau)\frac{d\tau}{\tau}=\frac{t^{2(n+\alpha)-5}}{(2(n+\alpha)-5)!}\equiv f_{\alpha n}(t)\quad(0\leqslant t\leqslant 1) \tag{3.3}$$

$$N_\alpha(y)=N_{11}(y)-(-1)^\alpha N_{12}(y)$$

$$N_{11}(y)=N_{22}(y)=\frac{1}{\pi}\left(\frac{1}{y-1}+\frac{1}{y+1}\right),$$

$$N_{12}(y)=N_{21}(y)=\frac{1}{\pi}\left(\frac{6y}{y^2+1}-\frac{4y^3}{(y^2+1)^2}\right)$$

In general, not every solution $\theta_{\alpha n}(\tau)$ of (3.3) will be a solution of (3.2) since it can yield a function different from $p_{\alpha n}(t)$ by an even polynomial of second degree in $d(t)$ on substituting into the left-hand side of (3.2) ($d(t)=d_0+d_1t^2$ for $\alpha=1$ and $d(t)=d_2t^2$ for $\alpha=2$). The polynomial $d(t)$ vanishes if the conditions

$$\int_0^1 L_1(0,\tau)\,\theta_{1n}(\tau)\,d\tau=\frac{2}{\pi}\int_0^1\tau^2\ln\tau\,\theta_{1n}(\tau)\,d\tau=p_{1n}(0) \tag{3.4}$$

$$\int_0^1\frac{\partial^2L_\alpha}{\partial t^2}(0,\tau)\,\theta_{\alpha n}(\tau)\,d\tau=\frac{2(3-\alpha)}{\pi}\int_0^1[1+(2-\alpha)\ln\tau]\,\theta_{\alpha n}(\tau)\,d\tau=p_{\alpha n}''(0)$$

are imposed on $\theta_{\alpha n}(\tau)$.

To satisfy them we seek the solution of (3.3) containing 3 - α arbitrary constants. We will solve (3.3) by the factorization method /1/ using a Mellin transform for which we write (3.3) in the form ($\psi_{\alpha n}(t)$ is an unknown function)

$$\int_0^\infty N\left(\frac{t}{\tau}\right)\theta_-(\tau)\frac{d\tau}{\tau}=f_-(t)+f_+(t)\quad(0\leqslant t<\infty) \tag{3.5}$$

$$\|\theta_-(t),f_-(t),f_+(t)\|=\begin{cases}\|\theta_{\alpha n}(t),f_{\alpha n}(t),0\| & (0\leqslant t\leqslant 1)\\ \|0,\ 0,\ \psi_{\alpha n}(t)\| & (1<t<\infty)\end{cases}$$

Applying the Mellin transformation to (3.5) and taking account of the formula in /18/ we obtain

$$G_\alpha(p)\,\Phi_{\alpha n}^-(p)=F^-(p)+F^+(p), \tag{3.6}$$
$$F^-(p)=[(p+2(n+\alpha)-5)(2(n+\alpha)-5)!]^{-1}$$

$$\|G_\alpha(p),\Phi_{\alpha n}^-(p),F^-(p),F^+(p)\|=\int_0^\infty\|N_\alpha(t),\theta_-(t),f_-(t),f_+(t)\|\,t^{p-1}\,dt \tag{3.7}$$

$$G_\alpha(p)=T(p)K_\alpha(p) \tag{3.8}$$
$$T(p)=\operatorname{tg}{}^1/_2\pi p,\quad K_\alpha(p)=1-(-1)^\alpha(2-p)/\sin{}^1/_2\pi p$$

The functions $\Phi_{\alpha n}^-(p)$, $F^-(p)$ are regular in $D^-=\{p:\max(-1,\operatorname{Re}\gamma)<\operatorname{Re}p\}$, γ is determined by the asymptotic form $\theta_{\alpha n}(\tau)=O(\tau^{-\gamma})$ as $\tau\to0$. The function $F^+(p)$ is regular in $D^+=\{p:\operatorname{Re}p<1\}$. The equality (3.6) is satisfied in the strip $\Omega=\{p:\max(-1,\operatorname{Re}\gamma)<\operatorname{Re}p<1\}$. Using standard reasoning /1/, we obtain the solution of problem (3.6) containing 3 - α arbitrary constants in the form

$$\Phi_{\alpha n}^-(p)=\{A_\alpha(p)+F^-(p)\,[G_\alpha^+(5-2(n+\alpha))]^{-1}\}\,[G_\alpha^-(p)]^{-1} \tag{3.9}$$
$$A_1(p)=a_0+a_1p,\quad A_2(p)=a_2$$
$$G_\alpha^\pm(p)=T^\pm(p)\,K_\alpha^\pm(p);\quad T^-(p)=\frac{T(p)}{T^+(p)},$$
$$T^+(p)=\frac{\Gamma(^1/_2-{}^1/_2p)}{\Gamma(1-{}^1/_2p)},\quad K_\alpha^-(p)=\frac{K_\alpha(p)}{K_\alpha^+(p)}$$
$$K_\alpha^+(p)=\exp\left(\frac{1}{2\pi i}\int_L\frac{\ln K_\alpha(q)}{q-p}\,dq\right)\begin{cases}(p-1)^{\alpha-1} & (\operatorname{Re}p\leqslant2)\\ (p-1)^{\alpha-1}K_\alpha(p) & (\operatorname{Re}p\geqslant2)\end{cases}$$

where L is the contour of $\operatorname{Re}q=2$. The selection of L instead of the contour L_1 with $\operatorname{Re}q\in(0,1)$ is because $\operatorname{Ind}K_\alpha=0$ on L while $\operatorname{Ind}K_2=-1$ on L_1. The presence of the zero for $K_2(p)$ for $p=1$ is taken into account by the factor $(p-1)$. As $p\to\infty$ $G_\alpha^\pm(p)=O(p^{\pm(\alpha-1/2)})$, $\Phi_{\alpha n}^-(p)=O(p^{1/2})$. The growth of $\Phi_{\alpha n}^-(p)$ as $p\to\infty$ is allowed when using the Mellin transform of generalized functions /19/ and indicates the presence of non-integrable singularities for the functions $\theta_{\alpha n}(\tau)$ as $\tau\to1-0$: $\theta_{\alpha n}(\tau)=O((1-\tau)^{-3/2})$ (similar to /2, 14-17/). The behaviour of $\theta_{\alpha n}(\tau)$ as $\tau\to+0$ is determined by the zeros of the function $G_\alpha(p)$ (see (1.22)) and agrees with the behaviour of the transverse forces at the vertices of a clamped quadrant /20/.

To find a_0, a_1, a_2, we use the fact that on the basis of (3.7) conditions (3.4) are carried over to the function $\Phi_{\alpha n}^-(p)$ in the following form

$$2\pi^{-1}(\Phi_{1n}^{-}(p))'_{p=3} = p_{1n}(0) \tag{3.10}$$
$$2(3-\alpha)\pi^{-1}(\Phi_{\alpha n}^{-}(p) + (2-\alpha)(\Phi_{\alpha n}^{-}(p))')_{p=1} = p''_{\alpha n}(0)$$
$$(\alpha = 1, 2)$$

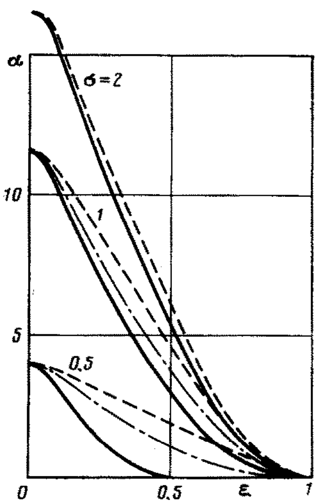
Substituting (3.9) into (3.10) and taking account of the known relationships (/21/ formula 1.7(1) and /18/, p.774), we obtain a system of linear algebraic equations in a_0, a_1, a_2 which completes the construction of the functions $\theta_{\alpha n}(\tau)$

$$\sum_{j=0}^{1} d_{ij}a_j = p_{1n}^{(2i)}(0)(4r)^{-1} - d_{i0}Q_1^-(p) - r(K_1^-(p))^{-1}Q_1^{-\prime}(p) \tag{3.11}$$
$$i = 0, 1;\ p = 3 - 2i,\ r = (2-i)^{-1},\ Q_\alpha^-(p) = F^-(p) \times [G_\alpha^+(5 - 2(n+\alpha))]^{-1}$$
$$d_{i0} = \frac{r}{K_1^-(p)}\left(-\frac{K_1^{-\prime}(p)}{K_1^-(p)} - \ln 2 + r\right), \quad d_{i1} = p\,d_{i0} + \frac{r}{K_1^-(p)}$$
$$(K_2^-(1))^{-1}(a_2 + Q_2^-(1)) = p_{2n}''(0)/2$$

Using the constructed $\theta_{\alpha n}(\tau)$, the approximate solution of system (3.1) is found from formulas (1.13).

It should be noted that the construction of the functions $\theta_{\alpha n}(\tau)$ can be considered as the solution of the problem of the bending of an infinite plate with a cruciform inclusion ($p_{\alpha n}(t)$ yield deflections on the inclusions). The solution of this problem is contained in the more general results of /22/, but it is obtained there in a form less convenient for realization of the scheme elucidated in Sect.1.

A relationship between the inclusion W_0 and the amount of force applied to the inclusion



$$P = \int_{-c_1}^{c_1} \psi_1(\xi)\,d\xi + \int_{-d_1}^{d_1} \psi_2(\eta)\,d\eta = \tag{3.12}$$
$$\frac{4}{\pi}\int_0^1 \varphi_1{}^\circ(\tau)\,d\tau =$$
$$\frac{4}{\pi}\sum_{n=0}^{N} \varphi_{1n}\Phi_{1n}^{-}(1)$$

is constructed from the results of the calculation.

The values of P for $N=2$ and $N=3$ are practically in agreement, which confirms the efficiency of the method proposed. The displacement of the inclusion was represented in the form

$$W_0 = 10^{-3}\alpha P a^2 D^{-1} \tag{3.13}$$

where the coefficient $\alpha = \alpha(\varepsilon, \sigma, \lambda)$ characterizes the stiffness of the plate-inclusion system. The values of α found at this section (for $\lambda = 1$) are presented as solid lines in the figure. We show for comparison, as dashed lines, the results for a linear inclusion located along the x axis, which corresponds to $\lambda = 0$ (see /2, 15/). As might have been expected, the cruciform inclusion is at a lower value. This decrease is especially noticeable for a plate stretched along the x axis ($\sigma = 0.5$), as well as for a square plate ($\sigma = 1$) for large inclusion dimensions ($\varepsilon \geqslant 0.8$). In the case of a plate stretched along the y axis ($\sigma = 2$) the influence of the second branch of the inclusion is unimportant.

4. Inclusion branches of different length ($\lambda \neq 1$). To be specific we consider $\lambda < 1$ (the x axis is directed along the long branch of the cross). System (2.8) is not reduced successfully to the form (1.12); consequently we will use (1.15) and (1.16). Let us construct the systems of solutions $\|\theta_{1\beta n}(\tau), \theta_{2\beta n}(\tau)\|$ of the characteristic system of equations with polynomials on the right-hand side

$$\frac{1}{2\pi}\sum_{j=1}^{2}\int_0^{\lambda_j} L_{ij}(t,\tau)\,\theta_{j\beta n}(\tau)\,d\tau = p_{i\beta n}(t) \quad (0 \leqslant t \leqslant \lambda_i,\ i = 1, 2) \tag{4.1}$$

$n = 0, 1, 2, \ldots, \beta = 1, 2$, the right-hand sides of the form $\|A + p(t), A\|$, $\beta = 2 - \|0, p(t)\|$ correspond to the value $\beta = 1$. It is here taken into account that the condition $p_{1\beta n}(0) = p_{2\beta n}(0)$ results from the behaviour of $L_{ij}(t,\tau)$ as $t \to 0$. Also taking into account that the evenness of $p_{i\beta n}(t)$ follows from the evenness of $L_{ij}(t,\tau)$, we take the following polynomials as $p_{i\beta n}(t)$:

$$p_{11n}(t)=\frac{t^{2n}}{(2n)!},\quad p_{21n}(t)=\delta_{0n},\quad p_{12n}(t)=0,\quad p_{22n}(t)=\frac{t^{2n+2}}{(2n+2)!} \tag{4.2}$$

Differentiating (4.1) thrice with respect to t, we obtain (see (3.3))

$$\sum_{j=1}^{2}\int_0^{\lambda_j} N_{ij}\left(\frac{t}{\tau}\right)\theta_{j\beta n}(\tau)\frac{d\tau}{\tau}=p_{i\beta n}'''(t)\equiv f_{i\beta n}(t)\quad (0\leqslant t\leqslant \lambda_i,\ i=1,2) \tag{4.3}$$
$$f_{i\beta n}(t) = \delta_{i\beta}t^{2(n+\beta)-5}\,[(2\,(n+\beta)-5)!]^{-1}$$

In order for the solution of the system (4.3) to be the solution of system (4.1) it should satisfy conditions analogous to (3.4)

$$\frac{1}{\pi}\sum_{j=1}^{2}\int_0^{\lambda_j}\tau^2\ln\tau\,\theta_{j\beta n}(\tau)\,d\tau=p_{1\beta n}(0) \tag{4.4}$$
$$\frac{1}{\pi}\sum_{j=1}^{2}\int_0^{\lambda_j}(2\ln\tau+1+2\delta_{ij})\,\theta_{j\beta n}(\tau)\,d\tau=p_{i\beta n}''(0)\quad (i=1,2)$$

We rewrite system (4.3) in the form ($\psi_{j\beta n}(t)$ is an unknown function)

$$\sum_{j=1}^{2}\int_0^{\infty} N_{ij}\left(\frac{t}{\tau}\right)\theta_{j-}(\tau)\frac{d\tau}{\tau}=f_{i-}(t)+f_{i+}(t)\quad (0\leqslant t<\infty,\ i=1,2) \tag{4.5}$$
$$\|\theta_{j-}(t), f_{j-}(t), f_{j+}(t)\|=\begin{cases}\|\theta_{j\beta n}(t), f_{j\beta n}(t), 0\| & (0\leqslant t\leqslant\lambda_j)\\ \|0, 0, \psi_{j\beta n}(t)\| & (\lambda_j<t<\infty)\end{cases}$$

Introducing the notation for the Mellin transform

$$\int_0^{\infty}\|N_{11}(t), N_{12}(t), \theta_{1-}(t), f_{1-}(t), f_{1+}(t)\|\,t^{p-1}\,dt= \tag{4.6}$$
$$\|T(p), R(p), \Phi_1^-(p), F_1^-(p), F_1^+(p)\|,\quad R(p)=(2-p)/\cos{}^1/_2\pi p$$
$$\int_0^{\infty}\|\theta_{2-}(t), f_{2-}(t), f_{2+}(t)\|\,t^{p-1}\,dt=\lambda^p\int_0^{\infty}\|\theta_{2-}(\lambda\xi), f_{2-}(\lambda\xi), f_{2+}(\lambda\xi)\|\xi^{p-1}d\xi=$$
$$\|\lambda^p\Phi_2^-(p), \lambda^pF_2^-(p), \lambda^pF_2^+(p)\|,\quad \Phi_j^-(p)\equiv\Phi_{j\beta n}^-(p)$$

and applying the Mellin transform to system (4.5), we obtain a Riemann matrix problem (the domains of regularity $D^\pm$ and the function $T(p)$ are the same as in (3.6)-(3.8)):

$$A_{11}(p)\Phi_1^-(p) + A_{12}(p)\lambda^p\Phi_2^-(p) = F_1^-(p) + F_1^+(p) \tag{4.7}$$
$$A_{21}(p)\Phi_1^-(p) + A_{22}(p)\lambda^p\Phi_2^-(p)=\lambda^pF_2^-(p)+\lambda^pF_2^+(p) \tag{4.8}$$
$$A_{11}(p) = A_{22}(p) = T(p),\ A_{12}(p) = A_{21}(p) = R(p) \tag{4.9}$$
$$F_i^-(p) = \delta_{i\beta}\lambda_\beta^{2(n+\beta)-5}\,[(p+2\,(n+\beta)-5)(2\,(n+\beta)-5)!]^{-1}\equiv F_{i\beta n}^-(p)$$

We will assume the presence of power-law singularities for the functions $\theta_{j-}(t)$ and $f_{j+}(t)$ as $t\to\lambda_j$. Then by theorems of Abelian type, their Mellin transforms $\Phi_j^-(p)(F_j^+(p))$ have a power-law behaviour as $p\to\infty, p\in D^-(D^+)$. The purpose of the subsequent constructions is the transformation of (4.7) and (4.8) to the form

$$C_i^+(p) = C_i^-(p)\ (i=1,2),\ C_i^\pm(p) = O(p^{\varkappa_i})\quad \text{as}\quad |p|\to\infty,\ p\in D^\pm \tag{4.10}$$

which affords the possiblity of applying Liouville's theorem.

We carry out the required transformation by replacing the specific functions (4.9) by arbitrary functions $A_{ij}(p)$ for generality, and assume here that

1) the functions $A_{12}(p)/A_{11}(p)$ $(A_{21}(p)/A_{11}(p))$ have only poles of multiplicity one p_m^- (p_m^+), $m=1,2,3,\ldots$, respectively, as singular points in $D^-(D^+)$ and behave as $O(p^{\mu_1})$ for $|p|\to\infty$, $p\in D_1(D_2)$, $D_{1,2}=\{p\in D^\mp, \inf|p-p_m^\mp|=\varepsilon_1>0\}$;

2) the factors $A_{11}^\pm(p)$, $G_{11}^\pm(p)$ behave as $O(p^{\mu_2})$ for $|p|\to\infty$, $p\in D^\pm$ for the

factorization $A_{11}(p) = A_{11}^{+}(p)A_{11}^{-}(p)$ and $G(p) \equiv A_{22}(p) - A_{12}(p)A_{21}(p)/A_{11}(p) = G^{+}(p)G^{-}(p)$;

3) $\operatorname{Re} p_m^{\mp} = O(m^{\alpha})$, $\alpha \geqslant 1$, $m \to \infty$;

4) $R_m^{\mp} = \operatorname*{Res}_{p=p_m^{\mp}} (A_{12,21}(p)/A_{11}(p)) = O(m^{\beta})$, $m \to \infty$.

For generality we write the functions $F_i^{-}(p)$ in the form $F_i^{-}(p) = b_i(p - a_i)^{-1}$, $a_i \in D^{+}$.

Since as $|p| \to \infty$, $p \in D^{+}$ the function $\lambda^{p} \to \infty$ more rapidly than any power of p, and components $A_{21}(p)\Phi_1^{-}(p)$ and $\lambda^{p}F_2^{+}(p)$ are simultaneously present in (4.8), then the reduction of (4.8) separately to the form (4.10) with the required asymptotic form is not possible. Consequently, we replace the equality (4.8) by the linear combination $-\lambda^{-p}A_{21}(p)/A_{11}(p)$ (4.7) + λ^{-p} (4.8):

$$G(p)\Phi_2^{-}(p) = -\lambda^{-p}\frac{A_{21}(p)}{A_{11}(p)}(F_1^{-}(p) + F_1^{+}(p)) + (F_2^{-}(p) + F_2^{+}(p)) \tag{4.11}$$

We rewrite the equalities (4.7) and (4.11) in the form

$$\begin{gathered}\frac{F_1^{-}(p) + F_1^{+}(p)}{A_{11}^{+}(p)} = A_{11}^{-}(p)\Phi_1^{-}(p) + \lambda^{p}\frac{A_{12}(p)}{A_{11}(p)}A_{11}^{-}(p)\Phi_2^{-}(p) \\ \frac{F_2^{-}(p) + F_2^{+}(p)}{G^{+}(p)} - \lambda^{-p}\frac{A_{21}(p)}{A_{11}(p)}\frac{(F_1^{-}(p) + F_1^{+}(p))}{G^{+}(p)} = G^{-}(p)\Phi_2^{-}(p)\end{gathered} \tag{4.12}$$

and introduce the following functions into the consideration

$$\begin{gathered}\Psi_1^{+}(p) = \sum_{m=1}^{\infty}\frac{\lambda^{p_m^{-}}A_m^{-}}{p - p_m^{-}}, \quad A_m^{-} = R_m^{-}A_{11}^{-}(p_m^{-})\Phi_2^{-}(p_m^{-}) \\ \Psi_2^{-}(p) = -\sum_{m=1}^{\infty}\frac{\lambda^{-p_m^{+}}A_m^{+}}{p - p_m^{+}}, \quad A_m^{+} = R_m^{+}\frac{F_1^{-}(p_m^{+}) + F_1^{+}(p_m^{+})}{G^{+}(p_m^{+})}\end{gathered} \tag{4.13}$$

whose subtraction from the corresponding components with the factors $\lambda^{\pm p}$ in (4.12) cancels the poles $p_m^{\pm}$ existing there. The series $\Psi_1^{+}(p)(\Psi_2^{-}(p))$ here converge uniformly in the domains $D_1(D_2)$, respectively, and behave as $O(p^{-1})$ as $|p| \to \infty$, $p \in D_1 \cup D^{+}$ $(p \in D_2 \cup D^{-})$. The numbers $A_m^{\pm}$ are still unknown since they are expressed in terms of the unknown functions $\Phi_2^{-}(p)$ and $F_1^{+}(p)$.

Taking account of the presence of the poles in D^{+} in (4.12), that are given by the functions $F_i^{-}(p)$, we introduce the notation

$$Q_1^{-}(p) = \frac{F_1^{-}(p)}{A_{11}^{+}(a_1)}, \quad Q_2^{-}(p) = \frac{F_2^{-}(p)}{G^{+}(a_2)} - F_1^{-}(p)\frac{\lambda^{-a_1}}{G^{+}(a_1)}\frac{A_{21}(a_1)}{A_{11}(a_1)} \tag{4.14}$$

Using the functions (4.13) and (4.14) we rewrite (4.12) in a form corresponding to (4.10):

$$\begin{aligned}C_1^{+}(p) &\equiv \frac{F_1^{-}(p) + F_1^{+}(p)}{A_{11}^{+}(p)} - Q_1^{-}(p) - \Psi_1^{+}(p) = \\ &A_{11}^{-}(p)\Phi_1^{-}(p) + \lambda^{p}\frac{A_{12}(p)}{A_{11}(p)}A_{11}^{-}(p)\Phi_2^{-}(p) - Q_1^{-}(p) - \\ &\Psi_1^{+}(p) \equiv C_1^{-}(p) \\ C_2^{+}(p) &\equiv \frac{F_2^{-}(p) + F_2^{+}(p)}{G^{+}(p)} - \lambda^{-p}\frac{A_{21}(p)}{A_{11}(p)}\frac{(F_1^{-}(p) + F_1^{+}(p))}{G^{+}(p)} - \\ &Q_2^{-}(p) - \Psi_2^{-}(p) = G^{-}(p)\Phi_2^{-}(p) - Q_2^{-}(p) - \Psi_2^{-}(p) \equiv C_2^{-}(p)\end{aligned} \tag{4.15}$$

The functions $C_i(p)$ mapped by the functions $C_i^{\pm}(p)$ in $D^{\pm}$, are entire and under the assumptions made regarding $A_{ij}(p)$ for sufficiently large p the $C_i(p)$ satisfy the inequalities $|C_i(p)| \leqslant B_i|p|^{\varkappa_i}$. According to Liouville's theorem $C_i(p) \equiv P_{\varkappa_i}(p)$ is equal to a polynomial of degree $\varkappa_i$. It is important to emphasize that since $\lambda^{p} \to 0$ as $|p| \to \infty$, $p \in D^{-}$, the component with factor λ^{p} present in $C_1^{-}(p)$ does not prevent satisfaction of the inequality mentioned. Moreover, since $C_1(p)$ is a polynomial, there are no components of the type $O(\lambda^{p}p^{\omega})$, in the asymptotic form $C_1^{-}(p)$, which can be explained by the presence of the asymptotic form $O(p^{\nu_1}) + O(\lambda^{p}p^{\nu_1})$ for the desired function $\Phi_1^{-}(p)$ and the mutual reduction of terms of the asymptotic form $O(\lambda^{p}p^{\omega})$ in the first two components in $C_1^{-}(p)$. All the above also refers to the component with the factor λ^{-p} in $C_2^{+}(p)$. Substituting $P_{\varkappa_i}(p)$ into (4.15) we obtain

$$\begin{gathered}F_1^{+}(p) = -F_1^{-}(p) + A_{11}^{+}(p)\Big(Q_1^{-}(p) + \sum_{l=1}^{\infty}\frac{\lambda^{p_l^{-}}A_l^{-}}{p - p_l^{-}} + P_{\varkappa_1}(p)\Big) \\ \Phi_2^{-}(p) = \frac{1}{G^{-}(p)}\Big(Q_2^{-}(p) - \sum_{l=1}^{\infty}\frac{\lambda^{-p_l^{+}}A_l^{+}}{p - p_l^{+}} + P_{\varkappa_2}(p)\Big)\end{gathered} \tag{4.16}$$

Substituting (4.16) into the expression for $A_m^{\mp}$ in (4.13), we arrive at an infinite system of linear algebraic equations in $A_m^{\mp}$

$$A_m^- = R_m^- \frac{A_{11}^-(p_m^-)}{G^-(p_m^-)}\Big(Q_2^-(p_m^-) - \sum_{l=1}^{\infty}\frac{\lambda^{-p_l^+}A_l^+}{p_m^- - p_l^+} + P_{\varkappa_2}(p_m^-)\Big) \tag{4.17}$$
$$A_m^+ = R_m^+ \frac{A_{11}^+(p_m^+)}{G^+(p_m^+)}\Big(Q_1^-(p_m^+) + \sum_{l=1}^{\infty}\frac{\lambda^{p_l^-}A_l^-}{p_m^+ - p_l^-} + P_{\varkappa_1}(p_m^+)\Big)$$

If we determine the polynomials $P_{\varkappa_i}(p)$ in some manner, then by solving system (4.17) (generally approximately), and substituting $A_m^{\pm}$ into (4.16), we find one of the solutions of problem (4.7), (4.8). For $P_{\varkappa_i}(p) \equiv 0$ we obtain a particular solution of the inhomogeneous problem. If we take $Q_i^-(p) \equiv 0$ and select $P_{\varkappa_1}(p) = p^k$ $(k = 0, 1, \ldots, \varkappa_1)$, $P_{\varkappa_2}(p) \equiv 0$ of $P_{\varkappa_1}(p) \equiv 0$, $P_{\varkappa_2}(p) = p^k$ $(k = 0, 1, \ldots, \varkappa_2)$, then we obtain a set $\varkappa_1 + \varkappa_2 + 2$ of solutions of the homogenous problem. As usual in the factorization method, the specific values of $\varkappa_1$ and $\varkappa_2$ are determined either by giving the behaviour of the solution $\theta_{j\beta n}(\tau)$ of the initial equations as $\tau \to \lambda_j - 0$ (and thereby giving the behaviour of $\Phi_j^-(p)$ as $|p| \to \infty$) by virtue of (4.6) and a theorem of Abelian type) or by the number of constants to satisfy some additional conditions. Three constants are required to satisfy conditions (4.4) in the case being considered here. Consequently, we take $\varkappa_1 = 1$, $\varkappa_2 = 0$, $P_{\varkappa_1}(p) = a_0 + a_1 p$, $P_{\varkappa_2}(p) = a_2$ (other modifications of the selection of $\varkappa_i$ that yield three arbitrary constants will result in divergence of the energy integral of a bent plate, see /17, 23/).

If $A_{ij}(p)$ and $F_i^-(p)$ are given by (4.9), we obtain

$$\frac{A_{12}(p)}{A_{11}(p)} = \frac{A_{21}(p)}{A_{11}(p)} = \frac{2-p}{\sin{}^1\!/_2\pi p}, \quad p_m^{\mp} = 2 \pm 2m$$
$$G(p) = T(p),\ K(p)\ K(p) = 1 - (p-2)^2/\sin^2{}^1\!/_2\pi p,\ G^{\pm}(p) = T^{\pm}(p)K^{\pm}(p)$$

The expressions for $T^{\pm}(p)$ and $K^{\pm}(p)$ are given by (3.9) with $K_2(p)$ replaced by $K(p)$.

Formulas (4.13), (4.14), (4.16) and (4.17) take the form

$$\Psi_1^+(p) = \sum_{m=1}^{\infty}\frac{\lambda^{2m+2}A_m^-}{p-2m-2}, \quad A_m^- = \frac{4}{\pi}(-1)^m m T^-(2+2m)\,\Phi_2^-(2+2m)$$
$$\Psi_2^-(p) = -\sum_{m=1}^{\infty}\frac{\lambda^{2m-2}A_m^+}{p+2m-2},$$
$$A_m^+ = -\frac{4}{\pi}(-1)^m m\,\frac{F_1^-(2-2m) + F_1^+(2-2m)}{G^+(2-2m)}$$
$$Q_1^-(p) = \frac{F_1^-(p)}{T^+(3-2n)}, \quad Q_2^-(p) = \frac{F_2^-(p)}{G^+(1-2n)} - \frac{F_1^-(p)}{\lambda^{3-2n}}\frac{(1-2n)(-1)^n}{G^+(3-2n)}$$
$$F_1^+(p) = -F_1^-(p) + T^+(p)\Big(Q_1^-(p) + \sum_{l=1}^{\infty}\frac{\lambda^{2l+2}A_l^-}{p-2l-2} + a_0 + a_1 p\Big)$$
$$\Phi_2^-(p) = \frac{1}{G^-(p)}\Big(Q_2^-(p) - \sum_{l=1}^{\infty}\frac{\lambda^{2l-2}A_l^+}{p+2l-2} + a_2\Big)$$
$$A_m^- + K_m^-\sum_{l=1}^{\infty}\frac{\lambda^{2l-2}A_l^+}{2(m+l)} = K_m^-(Q_2^-(2+2m) + a_2), \tag{4.18}$$
$$K_m^{\pm} = \frac{4m(-1)^m}{\pi K^{\pm}(2\mp 2m)}$$
$$A_m^+ - K_m^+\sum_{l=1}^{\infty}\frac{\lambda^{2l+2}A_l^-}{2(m+l)} = -K_m^+(Q_1^-(2-2m) + a_0 + (2-2m)a_1)$$

The infinite system (4.18) was solved by a reduction method for fixed a_i and the truncated system by a simple iteration method. From 10 equations for $\lambda = 0,2$ to 100 equations for $\lambda = 0.95$ were kept.

Note that the construction carried out are conceptually close to that used in /24/.

Let us determine the a_i by using conditions (4.4). To do this we execute a number of transformations. On the basis of (4.6) conditions (4.4) are carried over to the function $\Phi_i^-(p)$ in the following form (their sum and difference are taken in place of the last two equations in (4.4)):

$$\pi^{-1}(\Phi_1^-(p) + \lambda^p\Phi_2^-(p))'_{p=3} = p_{1\beta n}(0) \tag{4.19}$$
$$2(3-i)\pi^{-1}((2-i)(\Phi_1^-(p) \pm \lambda^p\Phi_2^-(p))' + (\Phi_1^-(p) \pm \lambda^p\Phi_2^-(p)))_{p=1} = p''_{1\beta n}(0) \pm p''_{2\beta n}(0) \quad (i = 1, 2)$$

In conformity with the structure of the right-hand sides in (4.18), the numbers $A_m^{\pm}$ depend linearly on a_i and the functions (4.13) can be written to the following form

$$\Psi_{1,2}^{\pm}(p) \equiv \Psi_{1,2}^{\pm}(p, a_0, a_1, a_2) = \Psi_{1,2}^{\pm}(p, 0, 0, 0) + a_0\Psi_{1,2}^{\pm}(p, 1, 0, 0) + a_1\Psi_{1,2}^{\pm}(p, 0, 1, 0) + a_2\Psi_{1,2}^{\pm}(p, 0, 0, 1) \tag{4.20}$$

An analogous representation can be written for the functions (4.16), and taking (4.7) and (4.8) into account for the functions

$$\Phi_1^-(p) \pm \lambda^p\Phi_2^-(p) = \pm\left(1 \mp \frac{R(p)}{T(p)}\right)\lambda^p\Phi_2^-(p) + \frac{F_1^-(p) + F_1^+(p)}{T(p)} \tag{4.21}$$

Substituting (4.21) into (4.19), we arrive at a system of three linear algebraic equations in a_i. **Furthermore,** the values of a_i are substituted into (4.18), and $A_m^{\pm}$ into (4.16), (4.7) and (4.8). Finally, we find the transforms $\lambda_j\Phi_j^-(p)$ of the functions $\theta_{j\beta n}(\tau)$ are solutions of system (4.1). The behaviour of the functions $\theta_{j\beta n}(\tau)$ as $\tau \to +0$ and $\tau \to \lambda_j - 0$ is the same as for the functions $\theta_{\alpha n}(\tau)$ in Sect.3. Using the constructed $\theta_{j\beta n}(\tau)$, we find the approximate solution of system (2.8) from (1.15) and (1.16).

As in Sect.3, the coefficient α in (3.13) is calculated from the results where

$$P = \frac{2}{\pi}\sum_{j=1}^{2}\int_0^{\lambda_j}\varphi_j(\tau)\,d\tau = \frac{2}{\pi}\sum_{j=1}^{2}\sum_{\beta=1}^{2}\sum_{n=0}^{N}\varphi_{\beta n}\lambda_j\Phi_j^-(1) \tag{4.22}$$

The values of P for $N = 2$ and $N = 3$ are practically in agreement.

For $\lambda < 1$ the case $\sigma < 1$ is of greatest interest. The corresponding values of α for $\sigma = 0.5$ and $\lambda = 0.5$ are presented as dash-dot lines in the left lower part of the figure. As might have been expected, the curve obtained lies in a domain bounded by the two limit curves $\lambda = 0$ and $\lambda = 1$. For a square plate $(\sigma = 1)$ the influence of the second branch of the inclusion is noticeable only for λ close to one (the appropriate dependence for $\lambda = 0.9$ is shown in the figure by dash-dots). In the case $\sigma = 2$ the influence of the second branch is unimportant and the appropriate results differ slightly from those presented in Sect.3. As in /2, 15/, for $\varepsilon = 0$ the values of α yield known values of the deflection under a concentrated force applied at the centre of a plate.

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THE PRESSURE OF A SYSTEM OF STAMPS ON AN ELASTIC HALF-PLANE UNDER GENERAL CONDITIONS OF CONTACT ADHESION AND SLIP*

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The contact interaction of an elastic half-plane and an arbitrary system of coupled and partially or completely detached stamps is considered. The problem is reduced to a combined Dirichlet-Riemann boundary value problem /1/ and is solved by quadratures. New modifications of the method and problems occurring in tasks with two and more slip sections are discussed; analogous problems with one slip section were studied earlier /2/. Fal'kovich's problem /3/ is investigated in a broadened formulation as an illustration.

1. Let $L_k = \langle a_k, b_k\rangle$, $k = 1, 2, \ldots, l$ be an open, half-open, or closed interval and $M_k = [p_k, q_k]$, $k = 1, 2, \ldots, m$, segments of the real axis $y = 0$ on which the stamps have, respectively, slipping contact and total adhesion with the elastic half-plane $-\infty < x < \infty, y \leqslant 0$; $a_1 < b_1 < \ldots < b_l$, $p_1 < q_1 < \ldots < q_m$. We determine the shape of the stamps, the tangential clearance on M_k, the separation-free abutment and non-intersection of the stamp and the half-plane by the boundary conditions

$$u' = u_0'(x), \quad x \in M; \quad v' = v_0'(x), \quad x \in L \cup M; \tag{1.1}$$
$$L = \bigcup_{k=1}^{l} L_k, \quad M = \bigcup_{k=1}^{m} M_k$$
$$\tau_{xy} = \tau_0(x), \ x \in L; \ \sigma_y = \tau_{xy} = 0, \ x \in S; \ L \cap M = 0$$
$$\sigma_y \leqslant 0, \ x \in L; \ v(x) - v_0(x) \geqslant 0, \ x \in S'$$

Here S is the **complement** $L \cup M$ to the real axis, S' are the selections outside $L \cup M$ on which the stamp base with the shape $v_0(x)$ is not contiguous to the half-plane; the given functions satisfy the Hölder condition; the interval $L_k = [a_k, b_k]$ $(L_k = (a_k, b_k))$ is

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